

SHIFTS OF FINITE TYPE AS FUNDAMENTAL OBJECTS IN THE THEORY OF SHADOWING

CHRIS GOOD AND JONATHAN MEDDAUGH

ABSTRACT. Shifts of finite type and the notion of shadowing, or pseudo-orbit tracing, are powerful tools in the study of dynamical systems. In this paper we prove that there is a deep and fundamental relationship between these two concepts.

Let X be a compact totally disconnected space and $f : X \rightarrow X$ a continuous map. We demonstrate that f has shadowing if and only if the system (f, X) is (conjugate to) the inverse limit of a directed system of shifts of finite type. In particular, this implies that, in the case that X is the Cantor set, f has shadowing if and only if (f, X) is the inverse limit of a sequence of shifts of finite type. Moreover, in the general compact metric case, where X is not necessarily totally disconnected, we prove that f has shadowing if and only if (f, X) is a factor of (i.e. semi-conjugate to) the inverse limit of a sequence of shifts of finite type by a quotient that almost lifts pseudo-orbits.

1. INTRODUCTION

Given a finite set of symbols, a shift of finite type consists of all infinite (or bi-infinite) symbol sequences, which do not contain any of a finite list of forbidden words, under the action of the shift map. Shifts of finite type have applications across mathematics, for example in Shannon's theory of information [24] and statistical mechanics. In particular, they have proved to be a powerful and ubiquitous tool in the study of hyperbolic dynamical systems. Adler and Weiss [1] and Sinai [29], for example, obtain Markov partitions for hyperbolic automorphisms of the torus and Anosov diffeomorphisms respectively, allowing analysis via shifts of finite type. Generalising the notion of Anosov diffeomorphisms, Smale [31] isolates subsystems conjugate to shifts of finite type in certain Axiom A diffeomorphisms. His fundamental example of a horseshoe, conjugate to the full shift space on two symbols, captures the chaotic behaviour of the diffeomorphism on the nonwandering set where the map exhibits hyperbolic behaviour. Bowen [5] then shows that the nonwandering set of any Axiom A diffeomorphism is a factor of a shift of finite type. In fact, shifts of finite type appear as horseshoes in many systems both hyperbolic (for example [32, 34]) and otherwise [16].

For a map f on a metric space X , a sequence $\langle x_i \rangle_{i \in \omega}$ is a δ -pseudo-orbit if $d(f(x_i), x_{i+1}) < \delta$. Pseudo-orbits arise naturally in the numerical calculation of orbits. It turns out that pseudo-orbits can often be tracked within a specified tolerance by real orbits, in which case f is said to have the shadowing, or pseudo-orbit

2000 *Mathematics Subject Classification.* 37B20, 54H20.

Key words and phrases. discrete dynamical system, inverse limits, pseudo-orbit tracing, shadowing, shifts of finite type.

The authors gratefully acknowledge support from the European Union through funding the H2020-MSCA-IF-2014 project ShadOmIC (SEP-210195797).

tracing, property. Clearly this is of importance when trying to model a system numerically (for example [8, 9, 18, 19]), especially when the system is expanding and errors might grow exponentially (indeed shadowing follows from expansivity for open maps [23], see also [21]). However, shadowing is also of theoretical importance and the notion can be traced back to the analysis of Anosov and Axiom A diffeomorphisms. Sinai [30] isolated subsystems of Anosov diffeomorphisms with shadowing and Bowen [4] proved explicitly that for the larger class of Axiom A diffeomorphisms, the shadowing property holds on the nonwandering set. However, Bowen [5] had already used shadowing implicitly as a key step in his proof that the nonwandering set of an Axiom A diffeomorphism is a factor of a shift of finite type. The notion of structural stability of a dynamical system was instrumental in the definitions of both Anosov and Axiom A diffeomorphisms [31] and shadowing plays a key role in stability theory [20, 22, 33]. Shadowing is also key to characterizing omega-limit sets [2, 4, 17]. Moreover, fundamental to the current paper is Walters' result [33] that a shift space has shadowing if and only if it is of finite type.

In this paper we prove that there is a deep and fundamental relationship between shadowing and shifts of finite type. It is known that shadowing is generic for homeomorphisms of the Cantor set [3] and that the shifts of finite type form a dense subset of the space of homeomorphisms on the Cantor set [25]. Hirsch [15] shows that expanding differentiable maps on closed manifolds are factors of the full one sided shift. In [5], Bowen considers the induced dynamics on the shift spaces associated with Markov partitions to show that the action of an Axiom A diffeomorphism on its non-wandering set is a factor of a shift of finite type. In this paper, we expand the scope of this type of analysis by considering the actions induced by f on shift spaces associated with several arbitrary finite open covers of the state space X , rather than the much more specific Markov partitions. In doing so, we are able to extend and clarify these results significantly. proving the following.

Theorem 17. *Let X be a compact, totally disconnected Hausdorff space. The map $f : X \rightarrow X$ has shadowing if and only if (f, X) is conjugate to the inverse limit of a directed system of shifts of finite type.*

Theorem 18. *Let X be the Cantor set, or indeed any compact, totally disconnected metric space. The map $f : X \rightarrow X$ has shadowing if and only if (f, X) is conjugate to the inverse limit of a sequence of shifts of finite type.*

Let X and Y be compact metric spaces and $\phi : X \rightarrow Y$ be a factor map (or semiconjugacy) between the systems $f : X \rightarrow X$ and $g : Y \rightarrow Y$ (so that $\phi(f(x)) = g(\phi(x))$). We say that ϕ almost lifts pseudo-orbits if and only if for all $\epsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ such that for any δ -pseudo-orbit $\langle y_i \rangle$ in Y , there exists an η -pseudo-orbit $\langle x_i \rangle$ in X such that $d(\phi(x_i), y_i) < \epsilon$.

Theorem 28. *Let X be a compact metric space. The map $f : X \rightarrow X$ has shadowing if and only if (f, X) is semiconjugate to the inverse limit of a sequence of shifts of finite type by a map which almost lifts pseudo-orbits.*

The approach we take is topological rather than metric as this seems to provide the most natural proofs and allows for simple generalization, at least in the zero-dimensional case. Although we are considering inverse limits of dynamical systems, our techniques are very similar in flavour to the inverse limit of coupled graph covers

which have been used by a number of authors to study dynamics on Cantor sets, for example [3, 10, 11, 13, 26, 27, 28].

The paper is arranged as follows. In Section 2, we formally define shadowing, shift of finite type and the inverse limit of a direct set of dynamical systems. In Section 3, we characterize shadowing as a topological, rather than metric property, and prove that an inverse limit of systems with shadowing itself has shadowing (Theorem 7). Here we also introduce the orbit and pseudo-orbit shift spaces associated with a finite open cover of a dynamical system and observe in Theorem 11 that these capture the dynamics of f . Section 4 discusses compact, totally disconnected Hausdorff, but not necessarily metric, dynamical systems, showing that such systems have shadowing of and only if they are (conjugate to) the inverse limit of a directed set of shifts of finite type (Theorem 17). In Section 5, we examine the case of general metric systems, establishing in Theorem 20 a partial analogue to Theorem 17. In Section 6, we discuss factor maps which preserve shadowing and, in light of this, we are able to completely characterize compact metric systems with shadowing in Theorem 28.

2. PRELIMINARIES AND DEFINITIONS

By map, we mean a continuous function. The set of natural numbers (including 0) is denoted by ω .

Definition 1. *Let X be a compact metric space and let $f : X \rightarrow X$ be a continuous function. Let $\langle x_i \rangle$ be a sequence in X . Then $\langle x_i \rangle$ is a δ -pseudo-orbit provided $d(x_{i+1}, f(x_i)) < \delta$ for all $i \in \omega$ and the point z ϵ -shadows $\langle x_i \rangle$ provided $d(x_i, f^i(z)) < \epsilon$ for all $i \in \omega$.*

The map f has shadowing (or the pseudo-orbit tracing property) provided that for all $\epsilon > 0$ there exists $\delta > 0$ such that every δ -pseudo-orbit is ϵ -shadowed by a point.

A particularly nice characterization of shadowing exists if we restrict our attention to *shift spaces*. For a finite set Σ , the *full one-sided shift with alphabet Σ* consists of the space of infinite sequence in Σ , i.e. Σ^ω using the product topology on the discrete space Σ and the *shift map* σ , given by

$$\sigma \langle x_i \rangle = \langle x_{i+1} \rangle.$$

A *shift space* is a compact invariant subset X of some full-shift. A shift space X is a *shift of finite type over alphabet Σ* if there is a finite collection \mathcal{F} of finite words in Σ for which $\langle x_i \rangle \in \Sigma^\omega$ belongs to X if and only if for all $i \leq j$, the word $x_i x_{i+1} \cdots x_j \notin \mathcal{F}$. A shift of finite type is said to be *N -step* provided that the length of the longest word in its associated set of forbidden words \mathcal{F} is $N + 1$. As mentioned above, a shift space has shadowing if and only if it is a shift of finite type [33].

Inverse limit constructions arise in a variety of settings. Many of the results here hold for arbitrary (non-metric) compact Hausdorff spaces and so we consider inverse limits of dynamical systems taken along an arbitrary directed set. The reader will not miss much by assuming that the space is compact metric in which case the inverse limit may be indexed by \mathbb{N} .

Definition 2. Let (Λ, \leq) be a directed set. For each $\lambda \in \Lambda$, let X_λ be a compact Hausdorff space and, for each pair $\lambda \leq \eta$, let $g_\lambda^\eta : X_\eta \rightarrow X_\lambda$ be a continuous surjection. Then $(g_\lambda^\eta, X_\lambda)$ is called an inverse system provided that

- (1) g_λ^λ is the identity map, and
- (2) for $\lambda \leq \eta \leq \nu$, $g_\lambda^\nu = g_\lambda^\eta \circ g_\eta^\nu$.

The inverse limit of $(g_\lambda^\eta, X_\lambda)$ is the space

$$\varprojlim \{g_\lambda^\eta, X_\lambda\} = \{\langle x_\lambda \rangle \in \prod X_\lambda : \forall \lambda \leq \eta, x_\lambda = g_\lambda^\eta(x_\eta)\}$$

with topology inherited as a subspace of the product $\prod X_\lambda$.

It is well known [12] that the inverse limit of compact Hausdorff spaces is itself compact and Hausdorff. Additionally, the following easily proved fact is often useful. If $U \subseteq \varprojlim \{g_\lambda^\eta, X_\lambda\}$ is open, and $x \in U$, then there exists λ and $U_\lambda \subseteq X_\lambda$ open with $x \in \pi_\lambda^{-1}(U_\lambda) \cap \varprojlim \{g_\lambda^\eta, X_\lambda\} \subseteq U$. That is, the collection of sets of the form $\pi_\lambda^{-1}(U_\lambda) \cap \varprojlim \{g_\lambda^\eta, X_\lambda\}$ for U_λ open in X_λ forms a basis for $\varprojlim \{g_\lambda^\eta, X_\lambda\}$.

Now, suppose that for each λ in the directed set Λ , $f_\lambda : X_\lambda \rightarrow X_\lambda$ is a continuous function. If the bonding maps g_λ^η commute with the functions f_λ , then we can extend this definition to the family of dynamical systems $\{(f_\lambda, X_\lambda) : \lambda \in \Lambda\}$. Specifically we make the following definition.

Definition 3. Let (Λ, \leq) be a directed set. For each $\lambda \in \Lambda$, let (f_λ, X_λ) be a dynamical system on a compact Hausdorff space and, for each pair $\lambda \leq \eta$, let $g_\lambda^\eta : X_\eta \rightarrow X_\lambda$ be a continuous surjection. Then $(g_\lambda^\eta, (f_\lambda, X_\lambda))$ is called an inverse system provided that

- (1) g_λ^λ is the identity map, and
- (2) for $\lambda \leq \eta \leq \nu$, $g_\lambda^\nu = g_\lambda^\eta \circ g_\eta^\nu$, and
- (3) for $\lambda \leq \eta$, $f_\lambda \circ g_\lambda^\eta = g_\lambda^\eta \circ f_\eta$ (i.e. that g_λ^η is a semiconjugacy).

The inverse limit of $(g_\lambda^\eta, (f_\lambda, X_\lambda))$ is the dynamical system $((f_\lambda)^*, \varprojlim \{g_\lambda^\eta, X_\lambda\})$, where $(f_\lambda)^*$ is the induced map given by

$$(f_\lambda)^*(\langle x_\lambda \rangle) = \langle f_\lambda(x_\lambda) \rangle.$$

That this defines a continuous dynamical system is a routine exercise. It is also immediate that if each of the maps f_λ is surjective, then the induced map $(f_\lambda)^*$ is also surjective.

3. SHADOWING WITHOUT METRICS

Shadowing is on first inspection a metric property, and indeed the properties of metrics often play a role in its investigation and application. However, we note in this section that shadowing can be viewed as a strictly topological property, provided that we restrict our attention to compact metric spaces. Similar observations have been made in [6, 14].

Definition 4. Let X be a space, let $f : X \rightarrow X$, and let \mathcal{U} be a finite open cover of X .

- (1) The sequence $\langle x_i \rangle_{i \in \omega}$ is a \mathcal{U} -pseudo-orbit provided for every $i \in \omega$, there exists $U_{i+1} \in \mathcal{U}$ with $x_{i+1}, f(x_i) \in U_{i+1}$. In this case, we say that the sequence $\langle U_i \rangle$ is a \mathcal{U} -pseudo-orbit pattern.

- (2) The point $z \in X$ \mathcal{U} -shadows $\langle x_i \rangle_{i \in \omega}$ provided for each $i \in \omega$ there exists $U_i \in \mathcal{U}$ with $x_i, f^i(z) \in U_i$. We call the sequence $\langle U_i \rangle$ an \mathcal{U} -orbit pattern.

Lemma 5. *Let X be a compact metric space. Then $f : X \rightarrow X$ has shadowing if and only if for every finite open cover \mathcal{U} , there exists a finite open cover \mathcal{V} , such that every \mathcal{V} -pseudo-orbit is \mathcal{U} -shadowed by some point $z \in X$.*

Proof. First, suppose that f has the shadowing property and let \mathcal{U} be a finite open cover of X . Fix $\epsilon > 0$ so that for each ϵ -ball B in X , there exists $U \in \mathcal{U}$ with $\overline{B} \subseteq U$. Now, let $\delta > 0$ witness ϵ -shadowing. Let \mathcal{V} be a finite open cover of X refining \mathcal{U} which consists of open sets of diameter less than δ .

Now, Let $\langle x_i \rangle$ be a sequence in X as in the statement of the lemma. Then $d(x_i, f(x_{i-1})) < \text{diam}(V_i) < \delta$ for all $i \in \omega \setminus 0$. In particular, $\langle x_i \rangle$ is a δ -pseudo-orbit. Let $z \in X$ be an ϵ -shadowing point for this sequence. Then $d(x_i, f^i(z)) < \epsilon$ for each $i \in \omega$, and in particular, $\{x_i, f^i(z)\} \subseteq B_\epsilon(x_i)$. By construction, there exists $U_i \in \mathcal{U}$ for which $\{x_i, f^i(z)\} \subseteq B_\epsilon(x_i) \subseteq U_i$, satisfying the conclusion of the lemma.

Conversely, let us suppose that f satisfies the open cover condition of the lemma. Let $\epsilon > 0$, and consider a finite subcover \mathcal{U} of X consisting of $\epsilon/2$ -balls. Let \mathcal{V} be the cover that witnesses the satisfaction of the condition, and choose $\delta > 0$ such that for each δ -ball in X , there is an element of \mathcal{V} which contains it.

Now, fix a δ -pseudo-orbit $\langle x_i \rangle$. Then for each $i \in \omega \setminus 0$, $d(x_i, f(x_{i-1})) < \delta$, and hence there exists $V_i \in \mathcal{V}$ such that $x_i, f(x_{i-1}) \in V_i$. Let $z \in X$ be the point guaranteed by the open cover condition. Then, for each $i \in \omega$, there exists $U_i \in \mathcal{U}$ with $x_i, f^i(z) \in U_i$. But U_i is an $\epsilon/2$ -ball and hence $d(x_i, f^i(z)) < \epsilon$, i.e. z ϵ -shadows the pseudo-orbit. \square

This observation allows the decoupling of shadowing from the metric, and we can then take the following definition of shadowing, which is valid for maps on any compact topological space and in particular for systems with compact Hausdorff domain, an application that has recently seen increased interest [7, 14].

Definition 6. *Let X be a (nonempty) topological space. The map $f : X \rightarrow X$ has shadowing provided that for every finite open cover \mathcal{U} , there exists a finite open cover \mathcal{V} such that every \mathcal{V} -pseudo-orbit is \mathcal{U} -shadowed by a point of X .*

With this definition in mind, we can prove the following result which will be important to the characterization of shadowing in Section 4.

Theorem 7. *Let $f : X \rightarrow X$ be conjugate to an inverse limit of maps with shadowing on compact spaces. Then f has shadowing.*

Proof. Without loss, let (Λ, \leq) be a directed set and $(f, X) = \varprojlim \{g_\gamma^\lambda, (f_\lambda, X_\lambda)\}$ where each of (f_λ, X_λ) is a system with shadowing on a compact space.

Let \mathcal{U} be a finite open cover of X . Since $X = \varprojlim \{g_\gamma^\lambda, X_\lambda\}$, we can find λ and a finite open cover \mathcal{W}_λ of X_λ so that $\mathcal{W}' = \{\pi_\lambda^{-1}(W) \cap X : W \in \mathcal{W}_\lambda\}$ refines \mathcal{U} . Since f_λ has shadowing, we can find a finite open cover \mathcal{V}_λ of X_λ witnessing this. Let $\mathcal{V} = \{\pi_\lambda^{-1}(V) \cap X : V \in \mathcal{V}_\lambda\}$.

Now, let $\langle x_i \rangle$ be a \mathcal{V} -pseudo-orbit with pattern $\langle \pi_\lambda^{-1}(V_i) \cap X \rangle$. Then for each $i \in \omega$, we have $f(\pi_\lambda^{-1}(V_i) \cap X) \cap \pi_\lambda^{-1}(V_{i+1}) \cap X \neq \emptyset$. It follows then, that $\langle (x_i)_\lambda \rangle$ is a \mathcal{V}_λ -pseudo-orbit with pattern $\langle V_i \rangle$. Since f_λ has shadowing, it follows then that

there is a sequence $\langle W_i \rangle$ in \mathcal{W} with $(x_i)_\lambda \in W_i$ and with $\bigcap f_\lambda^{-i}(\overline{W_i}) \neq \emptyset$. It then follows that $x_i \in \pi_\lambda^{-1}(W_i)$ and $\bigcap f^{-i}(\pi_\lambda^{-1}(W_i)) \neq \emptyset$, i.e. the pseudo-orbit $\langle x_i \rangle$ is \mathcal{W}' -shadowed, and hence \mathcal{U} -shadowed as required. \square

Now, the dynamics of the map f on X induce dynamics on the *space* of (pseudo-) orbit patterns. This is especially transparent if we take reasonably nice covers. In particular, we restrict our attention to *taut* covers, those for which the closure of elements meet only if the elements themselves meet.

Definition 8. Let $f : X \rightarrow X$ be a map on the space X , let \mathcal{U} be a taut open cover and let \mathcal{U}^ω be the one-sided shift space on the alphabet \mathcal{U} with shift map σ .

- (1) The \mathcal{U} -orbit space is the set $\mathcal{O}(\mathcal{U}) \subseteq \mathcal{U}^\omega$ consisting of all sequences $\langle U_i \rangle$ in \mathcal{U} for which there exists $z \in X$ with $f^i(z) \in \overline{U_i}$,
- (2) The \mathcal{U} -pseudo-orbit space is the set $\mathcal{PO}(\mathcal{U}) \subseteq \mathcal{U}^\omega$ consisting of all sequences $\langle U_i \rangle$ in \mathcal{U} for which there exists a sequence $\langle x_i \rangle$ with $x_{i+1}, f(x_i) \in \overline{U_{i+1}}$.

Additionally, for $U \in \mathcal{U}$ and $i \in \omega$, define $\pi_i : \mathcal{U}^\omega \rightarrow \mathcal{U}$ to be projection onto the i -th coordinate.

The following lemma is immediate and provides an alternate description of $\mathcal{O}(\mathcal{U})$ and $\mathcal{PO}(\mathcal{U})$.

Lemma 9. Let $f : X \rightarrow X$ be a map on X and let \mathcal{U} be a taut open cover. Then

$$\mathcal{O}(\mathcal{U}) = \{\langle U_i \rangle \in \mathcal{U}^\omega : \bigcap f^{-i}(\overline{U_i}) \neq \emptyset\}$$

and

$$\mathcal{PO}(\mathcal{U}) = \{\langle U_i \rangle \in \mathcal{U}^\omega : f(\overline{U_i}) \cap \overline{U_{i+1}} \neq \emptyset\}.$$

As consequence, we have the following relations between $\mathcal{O}(\mathcal{U})$, $\mathcal{PO}(\mathcal{U})$ and \mathcal{U}^ω .

Lemma 10. Let $f : X \rightarrow X$ be a map on X and let \mathcal{U} be a taut open cover. Then, $\mathcal{O}(\mathcal{U})$ is a subset of $\mathcal{PO}(\mathcal{U})$ and both spaces are subshifts of \mathcal{U}^ω . In particular, $\mathcal{PO}(\mathcal{U})$ is a 1-step shift of finite type.

Proof. That $\mathcal{O}(\mathcal{U}) \subseteq \mathcal{PO}(\mathcal{U}) \subseteq \mathcal{U}^\omega$ is immediate. It is also clear that all of these are shift invariant and closed, and hence are subshifts. That $\mathcal{PO}(\mathcal{U})$ is a 1-step shift of finite follows by observing that the condition that $f(\overline{U_i}) \cap \overline{U_{i+1}} \neq \emptyset$ is equivalent to forbidding words of the form UV where $f(\overline{U}) \cap \overline{V} = \emptyset$. \square

If X is compact Hausdorff, then the entire dynamics of a map f are encoded in the orbit spaces of an appropriate system of covers of X . In particular, let $\mathcal{TOC}(X)$ be the collection of all finite taut open covers of X . This collection is naturally partially ordered by refinement and forms a directed set.

Theorem 11. Let $f : X \rightarrow X$ be a map on the compact Hausdorff space X . Let $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ be a cofinal directed subset of $\mathcal{TOC}(X)$. Then for all $x \in X$ there exists a choice of $U_\lambda(x) \in \mathcal{U}_\lambda$ with $\{x\} = \bigcap \overline{U_\lambda(x)}$ and furthermore for any such sequence, we have for all $n \in \omega$,

$$\{f^n(x)\} = \bigcap \pi_0(\sigma^n(\mathcal{O}(\mathcal{U}_\lambda) \cap \pi_0^{-1}(\overline{U_\lambda(x)}))).$$

Proof. Let f , Λ , and $\{\mathcal{U}_\lambda\}$ be as described. Fix $x \in X$. For each $\lambda \in \Lambda$, choose $U_\lambda(x) \in \mathcal{U}_\lambda$ with $x \in U_\lambda(x)$. Then $x \in \bigcap \overline{U_\lambda(x)}$. Furthermore, for all $y \in X \setminus \{x\}$,

there exists a cover \mathcal{U} of X such that if $x \in U \in \mathcal{U}$, then $y \notin \overline{U}$. Then for any $\lambda \in \Lambda$ with \mathcal{U}_λ refining \mathcal{U} , $y \notin \overline{U_\lambda(x)}$ regardless of choice of $U_\lambda(x)$, and hence $y \notin \bigcap \overline{U_\lambda(x)}$.

Now, for each λ , it is straightforward to show that $\pi_0(\sigma^n(\mathcal{O}(\mathcal{U}_\lambda) \cap \pi_0^{-1}(\overline{U_\lambda(x)})))$ is equal to $f^n(\overline{U_\lambda(x)})$. In particular, $f^n(x) \in \bigcap \pi_0(\sigma^n(\mathcal{O}(\mathcal{U}_\lambda) \cap \pi_0^{-1}(\overline{U_\lambda(x)})))$. Suppose now that $z \in \bigcap \pi_0(\sigma^n(\mathcal{O}(\mathcal{U}_\lambda) \cap \pi_0^{-1}(\overline{U_\lambda(x)})))$. Then for each λ , there exists $x_\lambda \in U_\lambda(x)$ with $z = f^n(x_\lambda)$. But, by construction, x is a limit point of $\{x_\lambda\}$, and by continuity, $z = f^n(x)$. Hence $\{f^n(x)\} = \bigcap \pi_0(\sigma^n(\mathcal{O}(\mathcal{U}_\lambda) \cap \pi_0^{-1}(\overline{U_\lambda(x)})))$ as claimed. \square

It should be noted that for general Hausdorff spaces, the structure of $\{\mathcal{U}_\lambda\}$ may be quite complex, but for metric X , it is the case that a sequence of covers will always suffice and we will make use of this fact in the following sections. In the metric case, Theorem 11 is equivalent to Theorem 3.9 of [26], although that result is expressed in terms of graph covers and relations.

4. CHARACTERIZING SHADOWING IN TOTALLY DISCONNECTED SPACES

In the sense of Theorem 11, the entire dynamics are encoded by the action of f on an appropriate collection of refining covers. This is not unlike the way that the topology is completely encoded as well. In this section we explore this analogy.

In particular, it is well known that a space X is *chainable*, i.e. can be encoded with a sequence of refining *chains* (i.e. finite covers with $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$) if and only if X can be written as an inverse limit of arcs. In a sense, the arc is the *fundamental* chainable object. In an analogous fashion we show that shifts of finite type are the fundamental objects among dynamical systems on totally disconnected spaces with shadowing.

For a dynamical system (f, X) with X totally disconnected in addition to compact Hausdorff, then the covers \mathcal{U}_λ in Theorem 11 can be taken to consist of clopen sets which are pairwise disjoint. In this case, let \mathcal{U} and \mathcal{V} be finite clopen pairwise disjoint covers of X with \mathcal{V} refining \mathcal{U} . Then let $\iota : \mathcal{V} \rightarrow \mathcal{U}$ be defined so that $V \cap \iota(V) \neq \emptyset$, which in this context is equivalent to $V \subseteq \iota(V)$. Critically, for ι to be a function, the covers must consist of pairwise disjoint sets; it is this that creates the obstacle to dealing with non-totally disconnected spaces which we address in Section 5. The map ι naturally induces a continuous map $\iota : \mathcal{V}^\omega \rightarrow \mathcal{U}^\omega$, the domain of which can then be restricted to $\mathcal{O}(\mathcal{V})$ or $\mathcal{PO}(\mathcal{V})$ as appropriate. As the intended domain is typically clear, the symbol ι will be used for all.

Lemma 12. *Let $f : X \rightarrow X$ be a map on the compact, totally disconnected Hausdorff space X and let \mathcal{U} and \mathcal{V} be finite clopen pairwise disjoint covers of X with \mathcal{V} refining \mathcal{U} . Then σ and ι commute and the following statements hold:*

- (1) $\iota(\mathcal{O}(\mathcal{V})) = \mathcal{O}(\mathcal{U})$.
- (2) $\mathcal{O}(\mathcal{U}) \subseteq \iota(\mathcal{PO}(\mathcal{V})) \subseteq \mathcal{PO}(\mathcal{U})$.

Proof. It is immediate from their definitions that σ and ι commute on their unrestricted domains.

Towards proving statement (1), consider $\langle V_i \rangle \in \mathcal{O}(\mathcal{V})$. By Lemma 9, $\bigcap \overline{V_i} \neq \emptyset$, and it follows that $\bigcap \iota(\overline{V_i}) \neq \emptyset$, and hence $\langle \iota(V_i) \rangle \in \mathcal{O}(\mathcal{U})$. Conversely, if $\langle U_i \rangle \in \mathcal{O}(\mathcal{U})$, then we can choose $x \in \bigcap \overline{U_i} \neq \emptyset$. Now choose $\langle V_i \rangle$ so that $x \in \overline{V_i}$ for each i . Clearly $\langle V_i \rangle \in \mathcal{O}(\mathcal{V})$. Since \mathcal{V} refines \mathcal{U} and the elements of \mathcal{U} are pairwise disjoint and clopen, it follows that $V_i \subseteq U_i$, i.e. $\langle U_i \rangle = \iota(\langle V_i \rangle)$.

Statement (2) follows similarly. \square

The additional structure of totally disconnected spaces allows us to state the following immediate corollary to Theorem 11. In particular, the collection $\mathcal{PDO}\mathcal{C}(X)$ of finite pairwise disjoint covers is a cofinal directed subset of $\mathcal{TOC}(X)$.

Corollary 13. *Let $f : X \rightarrow X$ be a map on the compact Hausdorff totally disconnected space X . Let $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ be a cofinal directed suborder of $\mathcal{PDO}\mathcal{C}(X)$.*

Then the system (f, X) is conjugate to $(\sigma^, \varprojlim\{\iota, \mathcal{O}(\mathcal{U}_\lambda)\})$ by the map*

$$\langle w_\lambda \rangle \mapsto \bigcap \overline{\pi_0(w_\lambda)}.$$

It is important to note that the maps ι in the inverse system depend very much on their domain and range. However, if \mathcal{W} refines \mathcal{V} which in turn refines \mathcal{U} , then the composition of $\iota : \mathcal{W} \rightarrow \mathcal{V}$ and $\iota : \mathcal{V} \rightarrow \mathcal{U}$ is precisely the same as $\iota : \mathcal{W} \rightarrow \mathcal{U}$, and as such the inverse system is indeed well-defined.

The existence of pairwise disjoint covers also allows us to state the following alternative characterization of shadowing.

Lemma 14. *Let $f : X \rightarrow X$ be a map on the compact Hausdorff totally disconnected space X . Then f has shadowing if and only if for each $\mathcal{U} \in \mathcal{PDO}\mathcal{C}(X)$, there exists $\mathcal{V} \in \mathcal{PDO}\mathcal{C}(X)$ which refines \mathcal{U} such that for all $\mathcal{W} \in \mathcal{PDO}\mathcal{C}(X)$ which refine \mathcal{V} , $\iota(\mathcal{PO}(\mathcal{W})) = \mathcal{O}(\mathcal{U})$.*

Proof. Let f have shadowing and let $\mathcal{U} \in \mathcal{PDO}\mathcal{C}(X)$. Let \mathcal{V} be the cover witnessing shadowing. Without loss of generality, $\mathcal{V} \in \mathcal{PDO}\mathcal{C}(X)$. Now, let $\langle x_i \rangle$ be a \mathcal{V} -pseudo-orbit with $\langle V_i \rangle$ its \mathcal{V} -pseudo-orbit pattern, and let $z \in X$ be a shadowing point with $\langle U_i \rangle$ its shadowing pattern. By definition, x_i and $f^i(z)$ belong to U_i , and hence $V_i \cap U_i \neq \emptyset$, and since \mathcal{V} refines \mathcal{U} and the elements of \mathcal{U} are disjoint, we have $V_i \subseteq U_i$, i.e. $\iota(V_i) = U_i$ and hence $\iota\langle V_i \rangle = \langle U_i \rangle$. Thus $\iota(\mathcal{PO}(\mathcal{V})) \subseteq \mathcal{O}(\mathcal{U})$, and the reverse inclusion is given by Lemma 12, and thus the two sets are equal. Now, for any $\mathcal{W} \in \mathcal{PDO}\mathcal{C}(X)$ which refines \mathcal{V} , observe that $\mathcal{O}(\mathcal{U}) \subseteq \iota(\mathcal{PO}(\mathcal{W})) \subseteq \iota(\mathcal{PO}(\mathcal{V})) = \mathcal{O}(\mathcal{U})$.

Conversely, suppose that f has the stated property regarding open covers. Let \mathcal{U} be a finite open cover of X . Since X is totally disconnected, let $\mathcal{U}' \in \mathcal{PDO}\mathcal{C}(X)$ which refines \mathcal{U} . Let \mathcal{V} be the cover witnessing the property with respect to \mathcal{U}' . Now, let $\langle x_i \rangle$ be a \mathcal{V} -pseudo-orbit and let $\langle V_i \rangle$ be its \mathcal{V} -pseudo-orbit pattern. By the property, there exists $\langle U'_i \rangle \in \mathcal{O}(\mathcal{U}')$ with $\iota\langle V_i \rangle = \langle U'_i \rangle$. Now, let $z \in \bigcap f^{-i}(\overline{U'_i})$. Then $x_i, f^i(z) \in \overline{U'_i}$ which in turn is a subset of some $U_i \in \mathcal{U}$. In particular, z \mathcal{U} -shadows the \mathcal{V} -pseudo-orbit $\langle x_i \rangle$. \square

In light of this, we have the following theorem.

Theorem 15. *Let $f : X \rightarrow X$ be a map with shadowing on the compact totally disconnected Hausdorff space X . Let $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ be a cofinal directed suborder of $\mathcal{PDO}\mathcal{C}(X)$.*

Then the system $(\sigma^, \varprojlim\{\iota, \mathcal{O}(\mathcal{U}_\lambda)\})$ is conjugate to $(\sigma^*, \varprojlim\{\iota, \mathcal{PO}(\mathcal{U}_\lambda)\})$.*

Proof. For each $\lambda \in \Lambda$, let $p(\lambda) \geq \lambda$ such that $\iota(\mathcal{PO}(\mathcal{U}_{p(\lambda)})) = \mathcal{O}(\mathcal{U}_\lambda)$. Then, define the map $\phi : \varprojlim\{\iota, \mathcal{PO}(\mathcal{U}_\lambda)\} \rightarrow \varprojlim\{\iota, \mathcal{O}(\mathcal{U}_\lambda)\}$ as follows

$$\phi(\langle w_\gamma \rangle)_\lambda = \iota(w_{p(\lambda)})$$

That this is well-defined and continuous is a standard result in inverse limit theory. That it is a surjection follows from the surjectivity of ι from $\mathcal{PO}(\mathcal{U}_{p(\lambda)})$ onto $\mathcal{O}(\mathcal{U}_\lambda)$. As this is induced by the maps ι , it will commute with σ^* . All that remains is to demonstrate injectivity.

To this end, suppose that $\langle w_\gamma \rangle \neq \langle v_\gamma \rangle$ but $\phi(\langle w_\gamma \rangle) = \phi(\langle v_\gamma \rangle)$. Fix $\lambda \in \Lambda$ with $w_\lambda \neq v_\lambda$. Since these are elements of the inverse limit and the elements of \mathcal{U}_λ are disjoint, for all $\gamma \geq \lambda$, we have $w_\gamma \neq v_\gamma$. In particular, then, there is an $n \in \omega$ with $(w_\lambda)_n \neq (v_\lambda)_n$, and hence for all $\gamma \geq \lambda$, $(w_\gamma)_n \neq (v_\gamma)_n$. Thus the nested intersections of these are nonempty and distinct, i.e.

$$\emptyset \neq \bigcap_{\gamma \geq \lambda} \overline{(w_\gamma)_n} \neq \bigcap_{\gamma \geq \lambda} \overline{(v_\gamma)_n} \neq \emptyset.$$

This intersection may be taken over fewer sets, in particular we may restrict to those indices γ which are equal to $p(\eta)$ for some η , so we have

$$\emptyset \neq \bigcap_{\gamma \geq \lambda} \overline{(w_{p(\gamma)})_n} \neq \bigcap_{\gamma \geq \lambda} \overline{(v_{p(\gamma)})_n} \neq \emptyset.$$

On the other hand, since $\phi(\langle w_\gamma \rangle) = \phi(\langle v_\gamma \rangle)$, we have

$$\bigcap_{\gamma \geq \lambda} \overline{\iota(w_{p(\gamma)})_n} = \bigcap_{\gamma \geq \lambda} \overline{\iota(v_{p(\gamma)})_n}.$$

However all of these intersections are singletons and since $\iota(w_{p(\gamma)})_n \supseteq (w_{p(\gamma)})_n$, we have

$$\bigcap_{\gamma \geq \lambda} \overline{(w_{p(\gamma)})_n} = \bigcap_{\gamma \geq \lambda} \overline{\iota(w_{p(\gamma)})_n} = \bigcap_{\gamma \geq \lambda} \overline{\iota(v_{p(\gamma)})_n} = \bigcap_{\gamma \geq \lambda} \overline{(v_{p(\gamma)})_n},$$

which is a contradiction. □

This theorem complements Corollary 13, and by applying Lemma 10, and the well-known fact that shifts of finite type have shadowing [33], we have the following result.

Corollary 16. *Let $f : X \rightarrow X$ be a map with shadowing on the compact totally disconnected Hausdorff space X . Then (f, X) is conjugate to an inverse limit of shifts of finite type.*

In fact, this is a complete characterization of totally disconnected systems with shadowing; the following is an immediate consequence of Corollary 16 and Theorem 7.

Theorem 17. *Let X be a compact, totally disconnected Hausdorff space. The map $f : X \rightarrow X$ has shadowing if and only if (f, X) is conjugate to an inverse limit of shifts of finite type.*

Of course, Theorem 17 includes metric systems. However, if X is metric, we may easily find sequences $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ of finite pairwise disjoint open covers which are cofinal directed suborders of $\mathcal{PDO}(X)$. In particular, we can let $\mathcal{U}_0 = \{X\}$, and for each \mathcal{U}_i , let \mathcal{U}_{i+1} be a pairwise disjoint finite open cover of X with mesh less than 2^{-i} which refines \mathcal{U}_i and which witnesses shadowing for \mathcal{U}_i . Then the function p from the proof of Theorem 15 simply increments its input. The conjugacy then follows from the induced diagonal map ι^* on the inverse systems as seen in Figure 1.

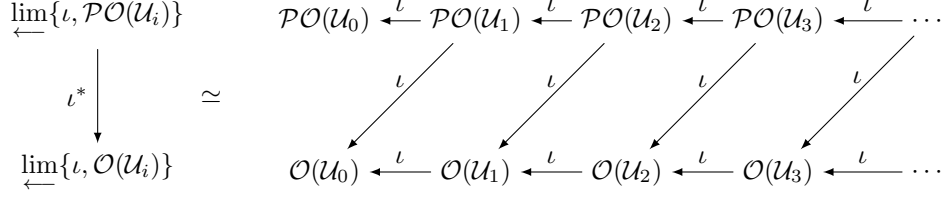


FIGURE 1. Diagram for the metric case of Theorem 17

This observation immediately implies the following.

Theorem 18. *Let X be the Cantor set, or indeed any compact, totally disconnected metric space. The map $f : X \rightarrow X$ has shadowing if and only if (f, X) is conjugate to the inverse limit of a sequence of shifts of finite type.*

This ad hoc construction of an appropriate sequence of covers can be modified into a technique that will apply to general compact metric spaces in Section 5.

We complete the section with another characterization of dynamical systems with shadowing on totally disconnected compact metric spaces. Once again, this result is analogous to a characterization of chainable continua in terms of ϵ -maps. Recall that an ϵ -map is a map $\phi : X \rightarrow Y$ such that for all $y \in Y$, $\phi^{-1}(y)$ has diameter less than ϵ . It is a routine exercise to prove that if $\phi : X \rightarrow Y$ is an ϵ -map and Y is compact metric, then for each $c > 0$, there exists $\eta > 0$ such that for all $U \subseteq Y$ with diameter less than η , $\phi^{-1}(U)$ has diameter less than $\epsilon + c$.

Theorem 19. *Let $f : X \rightarrow X$ be a map on the totally disconnected compact metric space X . Then f has shadowing if and only if for all $\epsilon > 0$, there exists a shift of finite type X_ϵ and an ϵ -map $\phi : X \rightarrow X_\epsilon$ such that ϕ is a semiconjugacy.*

Proof. First, observe that if f has shadowing, then it is conjugate to an inverse limit of a sequence of shifts of finite type as per the discussion following Theorem 17. In this case, with the usual metric on the product, the projection map onto the n -th factor space is a $\frac{1}{2^n}$ -map and is also a semiconjugacy as required.

Now, suppose that for each $\gamma > 0$, $f : X \rightarrow X$ is semiconjugate by a γ -map to a shift of finite type X_γ . Fix a particular $\epsilon > 0$ and let $\phi : X \rightarrow X_{\epsilon/2}$ be an $\epsilon/2$ -map and semiconjugacy. Now, find $\eta > 0$ so that if $U \subseteq X_{\epsilon/2}$ has diameter less than η , then the diameter of $\phi^{-1}(U)$ is less than ϵ . Since $X_{\epsilon/2}$ is a shift of finite type, the shift map has shadowing on $X_{\epsilon/2}$, so let us choose δ' so that every δ' -pseudo-orbit is η -shadowed. Finally, by uniform continuity of ϕ choose $\delta > 0$ so that if $d(x, y) < \delta$ that $d(\phi(x), \phi(y)) < \delta'$.

Now, let $\langle x_i \rangle$ be a δ -pseudo-orbit for f . Then $\langle \phi(x_i) \rangle$ is a δ' -pseudo-orbit for σ . As such, there exists $z' \in X_{\epsilon/2}$ which η -shadows it, in particular, the diameter of $\{\sigma^i(z'), \phi(x_i)\}$ is less than η , and so the diameter of $\phi^{-1}(\sigma^i(z')) \cup \{x_i\}$ is less than ϵ . Now, let $z \in \phi^{-1}(z')$. Then $f^i(z) \in \phi^{-1}(\sigma^i(z'))$, and so $d(f^i(z), x_i) < \epsilon$, i.e. z ϵ -shadows $\langle x_i \rangle$. \square

5. SHADOWING IN SYSTEMS WITH CONNECTED COMPONENTS

Theorem 11 applies equally well to systems in which there are non-trivial connected components, and as such, one might hope for analogue to Corollary 13.

However, as mentioned, the principal obstruction to a direct application of the methods of Section 4 is that the intersection relation ι is no longer necessarily single-valued, so that the induced map on the pseudo-orbit space is not only set-valued, but also not finitely determined. However, by modifying the approach illustrated in Figure 1, we obtain the following.

Theorem 20. *Let X be a compact Hausdorff space and $f : X \rightarrow X$ be continuous. Suppose that f has shadowing and that $\langle \mathcal{U}_i \rangle$ is a sequence of finite open covers satisfying the following properties:*

- (1) \mathcal{U}_{n+1} witnesses \mathcal{U}_n shadowing,
- (2) $\{\mathcal{U}_i\}$ is cofinal in $\mathcal{TOC}(X)$, and
- (3) for all $U \in \mathcal{U}_{n+2}$, there exists $W \in \mathcal{U}_n$ such that $st(U, \mathcal{U}_{n+1}) \subseteq W$.

Then there is an inverse sequence (g_n^{n+1}, X_n) of shifts of finite type such that (f, X) is semiconjugate to $(\sigma^, \varprojlim \{g_n^{n+1}, X_n\})$.*

Proof. Let $f : X \rightarrow X$ and covers $\langle \mathcal{U}_i \rangle$ be as stated. For each $U \in \mathcal{U}_{n+2}$, fix $W(U) \in \mathcal{U}_n$ with $st(U, \mathcal{U}_{n+1}) \subseteq W(U)$, and define $w : \mathcal{PO}(\mathcal{U}_{n+2}) \rightarrow \prod \mathcal{U}_n$ by $w(\langle U_j \rangle) = \langle W(U_j) \rangle$. Note that, as this is a single letter substitution map on a shift space, it is a continuous map and commutes with the shift map by definition.

We claim that $w(\mathcal{PO}(\mathcal{U}_{n+2}))$ is a subset of $\mathcal{O}(\mathcal{U}_n)$. Indeed, let $\langle U_j \rangle \in \mathcal{PO}(\mathcal{U}_{n+2})$ and $\langle x_j \rangle$ a pseudo-orbit with this pattern. Since \mathcal{U}_{n+2} witnesses \mathcal{U}_{n+1} -shadowing, there exists $z \in X$ and sequence $\langle V_j \rangle \in \mathcal{O}(\mathcal{U}_{n+1})$ with $f^j(z), x_j \in V_j$. In particular, for any such z and choices of $\langle V_j \rangle$, $V_j \subseteq st(U_j, \mathcal{U}_{n+1}) \subseteq W(U_j)$. Indeed, this establishes that $\langle W(U_j) \rangle \in \mathcal{O}(\mathcal{U}_n)$. It should be noted that while w is not necessarily surjective, for every $x \in X$, there is some $\langle U_j \rangle$ in $w(\mathcal{PO}(\mathcal{U}_{n+2}))$ with $f^j(x) \in U_j$ for all j . We can observe this by noting that $\langle f^j(x) \rangle$ is itself a \mathcal{U}_{n+2} -pseudo-orbit, and in particular, we have $f^j(x) \in W(U_j)$, and so $\langle W(U_j) \rangle$ is a \mathcal{U}_n -orbit pattern for x .

Since $\mathcal{O}(\mathcal{U}_n) \subseteq \mathcal{PO}(\mathcal{U}_n)$, we have the following diagram.

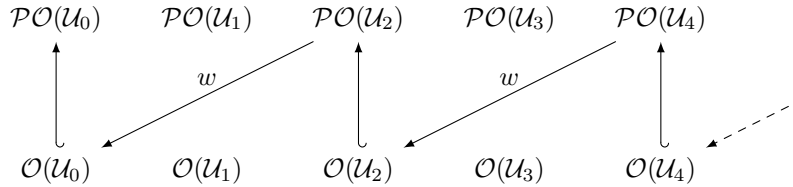


FIGURE 2. Diagram for the proof of Theorem 20

So, while the ‘natural’ map from $\mathcal{PO}(\mathcal{U}_{n+2})$ is set-valued, the composition of inclusion and w gives a single-valued continuous map from $\mathcal{PO}(\mathcal{U}_{n+2})$ to $\mathcal{PO}(\mathcal{U}_n)$, and by reversing the order of composition, from $\mathcal{PO}(\mathcal{U}_{n+2})$ to $\mathcal{PO}(\mathcal{U}_n)$. We will denote these maps by ι' . Figure 3 then establishes the conjugacy between the inverse limits of the pseudo-orbit spaces and orbit spaces. Once again, while these maps are not surjective, it is important to note that they do preserve much of the structure. In particular, for each $x \in X$, and each $n \in \mathbb{N}$ and $k \in \omega$, there is some element of $w \circ (\iota')^n(\mathcal{PO}(\mathcal{U}_{2(n+1)+2k}))$ which is a \mathcal{U}_{2k} -orbit pattern for x . This follows immediately from the comments preceding Figure 2.

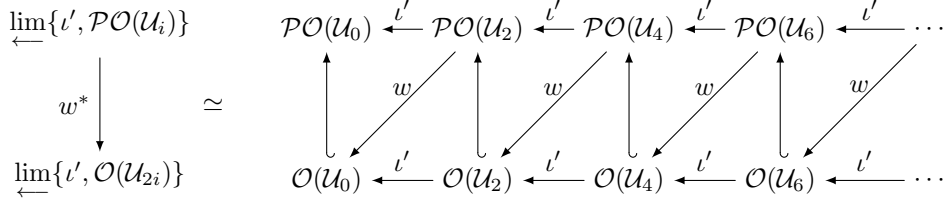


FIGURE 3. Diagram for the proof of Theorem 20

All that remains is to establish that the inverse limit of orbit spaces is semi-conjugate to the system (f, X) . Let $\phi : \varprojlim \{\iota', \mathcal{O}(\mathcal{U}_{2i})\} \rightarrow X$ be given by $\phi\langle u_i \rangle = \bigcap \overline{\pi_0(u_i)}$. Note that, by construction, $\pi_k(u_{i+1}) \subseteq \pi_k(u_i)$ for all k and i , so in particular $\phi(\langle u_i \rangle)$ is a nested intersection of the closures of elements of the open covers, and hence is well-defined. That ϕ is continuous and commutes with σ^* follows from similar reasoning as Theorem 11 and Corollary 13. All that remains is to establish that ϕ is surjective, but this follows from the aforementioned fact that for all $x \in X$, and all k, n , there exists a \mathcal{U}_{2k} -orbit pattern for x in $w \circ (\iota')^n(\mathcal{PO}(\mathcal{U}_{2(n+1)+2k}))$. \square

Clearly, the existence of a sequence of covers satisfying conditions (2) and (3) of Theorem 20 is a strong condition. In particular, the fact that this sequence is cofinal in $\mathcal{TOC}(X)$ implies that the space X is metrizable. In fact, it is the case that for all compact metric spaces, conditions (2) and (3) are easily satisfiable, yielding the following.

Corollary 21. *Let $f : X \rightarrow X$ be a map with shadowing on the compact metric space X . Then there is an inverse sequence (g_n^{n+1}, X_n) of shifts of finite type such that (f, X) is semiconjugate to $(\sigma^*, \varprojlim \{g_n^{n+1}, X_n\})$.*

Proof. By Theorem 20, we need only demonstrate the existence of a sequence of covers satisfying (1), (2), and (3). This is easily accomplished by taking $\mathcal{U}_0 = \{X\}$, and inductively letting \mathcal{U}_{n+1} be a cover witnessing \mathcal{U}_n -shadowing with mesh less than one third the Lebesgue number of the cover \mathcal{U}_n . Conditions (1) and (2) are immediately met. To verify that condition (3) is satisfied, fix $n \in \mathbb{N}$ and $U \in \mathcal{U}_{n+2}$. Then U is a subset of V for some $V \in \mathcal{U}_{n+1}$, and so $st(U, \mathcal{U}_{n+1})$ is a subset of $st(V, \mathcal{U}_{n+1})$. But the diameter of $st(V, \mathcal{U}_{n+1})$ is at most three times the mesh of \mathcal{U}_{n+1} , and hence has diameter less than the Lebesgue number for \mathcal{U}_n . Hence there is some $W \in \mathcal{U}_n$ for which $W \supseteq st(V, \mathcal{U}_{n+1}) \supseteq st(U, \mathcal{U}_{n+1})$ as required. \square

6. FACTOR MAPS WHICH PRESERVE SHADOWING

We have now established that the existence of a semiconjugate inverse limit of an inverse sequence of shifts of finite type is necessary for a metric system to exhibit shadowing. However, it is worth noting that this is by no means sufficient. In particular, every sofic shift is semiconjugate to such an inverse limit, but only those that are shifts of finite type exhibit shadowing.

Remark 22. Let X be the subshift of $\{0, 1\}^{\mathbb{Z}}$ consisting of those bi-infinite words containing at most one 1. The system (σ, X) fails to have shadowing, but is semiconjugate to the inverse limit of an inverse sequence of shifts of finite type.

Proof. Let Y be the subshift of $\{0, 1, 2\}^{\mathbb{Z}}$ consisting of those sequences in which the words 02, 10, 21 and 20 do not appear, i.e. Y is the subshift of all sequences of the form $\dots 000000122222\dots$ along with the constant sequences $\langle 0 \rangle$ and $\langle 2 \rangle$. Y is a shift of finite type. Then Y is (trivially) an inverse limit of shifts of finite type. However, the map from Y to X induced by substituting the symbol 0 for 2 is a semiconjugacy. \square

It is then natural to ask which factors of inverse limits of shifts of finite type exhibit shadowing, i.e. is there a class of maps \mathcal{P} such that if (f, X) is a factor by a map in \mathcal{P} of an inverse limit of shifts of finite type, then (f, X) has shadowing? Of course, it is clear that there is such a class, and in fact the class of homeomorphisms have this property, but we wish to find, if possible, the maximal such class. Towards this end, we define the following.

Definition 23. Let (f, X) and (g, Y) be dynamical systems with X and Y compact Hausdorff spaces. A semiconjugacy $\phi : (f, X) \rightarrow (g, Y)$ lifts pseudo-orbits provided that for every $\mathcal{V}_X \in \mathcal{TOC}(X)$, there exists $\mathcal{V}_Y \in \mathcal{TOC}(Y)$ such that if $\langle y_i \rangle$ is a \mathcal{V}_Y -pseudo-orbit in Y , then there is a \mathcal{V}_X -pseudo-orbit $\langle x_i \rangle$ in X with $\langle y_i \rangle = \langle \phi(x_i) \rangle$.

Theorem 24. Let (f, X) and (g, Y) be dynamical systems with X and Y compact Hausdorff. If (f, X) has shadowing and $\phi : (f, X) \rightarrow (g, Y)$ is a semiconjugacy that lifts pseudo-orbits, then (g, Y) has shadowing.

Proof. Fix an open cover $\mathcal{U}_Y \in \mathcal{TOC}(Y)$, and let $\mathcal{U}_X \in \mathcal{TOC}(X)$ such that $\phi(\mathcal{U}_X)$ refines \mathcal{U}_Y . Since (f, X) has shadowing, let $\mathcal{V}_X \in \mathcal{TOC}(X)$ witness shadowing with respect to \mathcal{U}_X .

Since ϕ lifts pseudo-orbits, let \mathcal{V}_Y witness this with respect to \mathcal{V}_X . Finally, let $\langle y_i \rangle$ be a \mathcal{V}_Y -pseudo-orbit.

Pick $\langle x_i \rangle$ to be a \mathcal{V}_X -pseudo-orbit with $\langle \phi(x_i) \rangle = \langle y_i \rangle$. As every \mathcal{V}_X -pseudo-orbit is \mathcal{U}_X -shadowed, fix $z_X \in X$ to witness this and let $z_Y = \phi(z_X)$. It then follows that for each i , we have $\phi^i(z_X), \phi(x_i) \in \phi(U_{X,i})$ for some $U_i \in \mathcal{U}_X$. As $\phi(\mathcal{U}_X)$ refines \mathcal{U}_Y , it follows that there exists $U_{Y,i} \in \mathcal{U}_Y$ with $\phi^i(z_Y), y_i \in \phi(U_{Y,i})$, i.e. $z_Y = \phi(z_X)$ \mathcal{U}_Y -shadows $\langle y_i \rangle$. \square

Notwithstanding Theorem 24, a more general concept of lifting pseudo-orbits provides a much sharper insight into the relation between shadowing and shifts of finite type in compact metric systems.

Definition 25. Let (f, X) and (g, Y) be dynamical systems with X and Y compact Hausdorff spaces. A semiconjugacy $\phi : (f, X) \rightarrow (g, Y)$ almost lifts pseudo-orbits (or f is an ALP map) provided that for every $\mathcal{V}_X \in \mathcal{TOC}(X)$ and $\mathcal{W}_Y \in \mathcal{TOC}(Y)$, there exists $\mathcal{V}_Y \in \mathcal{TOC}(Y)$ such that if $\langle y_i \rangle$ is a \mathcal{V}_Y -pseudo-orbit in Y , then there is a \mathcal{V}_X -pseudo-orbit $\langle x_i \rangle$ in X such that for each $i \in \mathbb{N}$ there exists $W_i \in \mathcal{W}_Y$ with $\phi(x_i), y_i \in W_i$.

Theorem 26. Let (f, X) and (g, Y) be dynamical systems with X and Y compact Hausdorff and let $\phi : (f, X) \rightarrow (g, Y)$ be a semiconjugacy. Then the following statements hold:

- (1) if (f, X) has shadowing and ϕ is an ALP map, then (g, Y) has shadowing, and
- (2) if (g, Y) has shadowing then ϕ is an ALP map.

Proof. First, we prove statement (1). Let (f, X) have shadowing and let ϕ be an ALP map. Fix an open cover $\mathcal{U}_Y \in \mathcal{TOC}(Y)$. Let $\mathcal{W}_Y \in \mathcal{TOC}(Y)$ such that if $W, W' \in \mathcal{W}_Y$ with $W \cap W' \neq \emptyset$, then there exists $V \in \mathcal{V}_Y$ with $W \cup W' \subseteq V$, and let $\mathcal{U}_X \in \mathcal{TOC}(X)$ such that $\phi(U_X)$ refines \mathcal{W}_Y . Since (f, X) has shadowing, let $\mathcal{V}_X \in \mathcal{TOC}(X)$ witness shadowing with respect to \mathcal{U}_X .

Since ϕ is ALP, let \mathcal{V}_Y witness this with respect to \mathcal{W}_Y and \mathcal{V}_X . Finally, let $\langle y_i \rangle$ be a \mathcal{V}_Y -pseudo-orbit.

Pick $\langle x_i \rangle$ to be a \mathcal{V}_X -pseudo-orbit so that $\langle \phi(x_i) \rangle$ \mathcal{W}_Y -shadows $\langle y_i \rangle$. As every \mathcal{V}_X -pseudo-orbit is \mathcal{U}_X -shadowed, fix $z_X \in X$ to witness this and let $z_Y = \phi(z_X)$. It then follows that for each i , we have $\phi^i(z_X), \phi(x_i) \in \phi(U_{X,i})$ for some $U_i \in \mathcal{U}_X$. As $\phi(U_X)$ refines \mathcal{W}_Y , it follows that there exists $W_i \in \mathcal{W}_Y$ with $\phi^i(z_Y), \phi(x_i) \in W_i$. Additionally, as pseudo-orbits are almost lifted, there exists $W'_i \in \mathcal{W}_Y$ with $\phi(x_i), y_i \in W'_i$. In particular, $\phi(x_i) \in W_i \cap W'_i$, so we have that there exists $U_{Y,i} \in \mathcal{U}_Y$ with $\phi^i(z_Y), \phi(x_i), y_i \in W_i \cup W'_i \subseteq U_{Y,i} \in \mathcal{U}_Y$, i.e. z_Y \mathcal{U}_Y -shadows $\langle y_i \rangle$.

Now, to prove statement (2), assume that (g, Y) has shadowing. let $\mathcal{V}_X \in \mathcal{TOC}(X)$ and $\mathcal{W}_Y \in \mathcal{TOC}(Y)$. Let $\mathcal{V}_Y \in \mathcal{TOC}(Y)$ witness shadowing with respect to \mathcal{W}_Y . Now, let $\langle y_i \rangle$ be a \mathcal{V}_Y pseudo-orbit in Y . Then there is some $z \in Y$ with z \mathcal{W}_Y -shadowing $\langle y_i \rangle$. Now, choose $x \in \phi^{-1}(z)$ and observe that $\langle f^i(x) \rangle$ is a \mathcal{V}_X -pseudo-orbit (as it is in fact an orbit), and $\phi(f^i(x)) = g^i(z)$, so there exists $W_i \in \mathcal{W}_Y$ with $\phi(f^i(z)), y_i \in W_i$ (as given by the shadowing pattern of z and $\langle y_i \rangle$). Thus ϕ is an ALP map. \square

One consequence of the above is that the (semi-)conjugacies in Theorems 17 and 20 and in Corollary 21 are ALP maps. This is not terribly surprising in the case of Theorem 17, as the map in question is a homeomorphism, but in the case of Theorem 20 and Corollary 21, this is interesting, and allows us to complete the classifications. It should be also be noted that, as a result of this theorem, we see that the factor maps constructed by Bowen in [5] are ALP.

To complete the characterization, we first translate the notion of almost lifting pseudo-orbits from the language of covers into the language of metric spaces. This is not completely necessary, but allows for a different perspective on the property. As this is a direct translation, we state the following without proof.

Lemma 27. *Let (f, X) and (g, Y) be dynamical systems with X and Y compact metric spaces. A semiconjugacy $\phi : (f, X) \rightarrow (g, Y)$ is an ALP map if and only if for all $\epsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ such that if $\langle y_i \rangle$ is a δ -pseudo-orbit in Y , there exists an η -pseudo-orbit $\langle x_i \rangle$ in X with $d(\phi(x_i), y_i) < \epsilon$.*

Theorem 28. *Let X be a compact metric space. The map $f : X \rightarrow X$ has shadowing if and only if it is semiconjugate to the inverse limit of a sequence of shifts of finite type by a map which is ALP.*

Proof. This follows immediately from Corollary 21 and part (2) of Theorem 26. \square

Of course, it would be of significant benefit if ALP maps had an alternate characterization. In particular, it is clear that homeomorphisms and covering maps lift pseudo-orbits. However, there are maps which are neither covering maps nor homomorphisms which almost lift pseudo-orbits, in particular, the semiconjugacies

given in Theorem 20 and Corollary 21 are not typically open, much less covering maps.

REFERENCES

- [1] R. L. Adler and B. Weiss. Entropy, a complete metric invariant for automorphisms of the torus. *Proc. Nat. Acad. Sci. U.S.A.*, 57:1573–1576, 1967.
- [2] A. D. Barwell, C. Good, P. Oprocha, and B. E. Raines. Characterizations of ω -limit sets of topologically hyperbolic spaces. *Discrete Contin. Dyn. Syst.*, 33(5):1819–1833, 2013.
- [3] Nilson C. Bernardes, Jr. and Udayan B. Darji. Graph theoretic structure of maps of the Cantor space. *Adv. Math.*, 231(3-4):1655–1680, 2012.
- [4] R. Bowen. ω -limit sets for axiom A diffeomorphisms. *J. Differential Equations*, 18(2):333–339, 1975.
- [5] Rufus Bowen. Markov partitions for Axiom A diffeomorphisms. *Amer. J. Math.*, 92:725–747, 1970.
- [6] W. R. Brian. Abstract omega-limit sets.
- [7] W. R. Brian. Ramsey shadowing and minimal points. *Proc. Amer. Math. Soc.*, 144(6):2697–2703, 2016.
- [8] R. M. Corless and S. Y. Pilyugin. Approximate and real trajectories for generic dynamical systems. *J. Math. Anal. Appl.*, 189(2):409–423, 1995.
- [9] Robert M. Corless. Defect-controlled numerical methods and shadowing for chaotic differential equations. *Phys. D*, 60(1-4):323–334, 1992. Experimental mathematics: computational issues in nonlinear science (Los Alamos, NM, 1991).
- [10] Alexandre I. Danilenko. Strong orbit equivalence of locally compact Cantor minimal systems. *Internat. J. Math.*, 12(1):113–123, 2001.
- [11] Tomasz Downarowicz and Alejandro Maass. Finite-rank Bratteli-Vershik diagrams are expansive. *Ergodic Theory Dynam. Systems*, 28(3):739–747, 2008.
- [12] Ryszard Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.
- [13] Leobardo Fernandez, Chris Good, and Mate Puljiz. Almost minimal systems and periodicity in hyperspaces. *Ergodic Theory Dynam. Systems*.
- [14] C. Good and S. Macias. What is topological about topological dynamics.
- [15] Morris W. Hirsch. Expanding maps and transformation groups. In *Global analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, pages 125–131. Amer. Math. Soc., Providence, R.I., 1970.
- [16] Judy Kennedy and James A. Yorke. Topological horseshoes. *Trans. Amer. Math. Soc.*, 353(6):2513–2530, 2001.
- [17] Jonathan Meddaugh and Brian E. Raines. Shadowing and internal chain transitivity. *Fund. Math.*, 222(3):279–287, 2013.
- [18] Ken Palmer. *Shadowing in dynamical systems*, volume 501 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2000. Theory and applications.
- [19] D. W. Pearson. Shadowing and prediction of dynamical systems. *Math. Comput. Modelling*, 34(7-8):813–820, 2001.
- [20] S. Y. Pilyugin. *Shadowing in dynamical systems*, volume 1706 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999.
- [21] Feliks Przytycki and Mariusz Urbański. *Conformal fractals: ergodic theory methods*, volume 371 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2010.
- [22] Clark Robinson. Stability theorems and hyperbolicity in dynamical systems. In *Proceedings of the Regional Conference on the Application of Topological Methods in Differential Equations (Boulder, Colo., 1976)*, volume 7, pages 425–437, 1977.
- [23] K. Sakai. Various shadowing properties for positively expansive maps. *Topology Appl.*, 131(1):15–31, 2003.
- [24] C. E. Shannon. A mathematical theory of communication. *Bell System Tech. J.*, 27:379–423, 623–656, 1948.
- [25] Takashi Shimomura. The pseudo-orbit tracing property and expansiveness on the Cantor set. *Proc. Amer. Math. Soc.*, 106(1):241–244, 1989.

- [26] Takashi Shimomura. Special homeomorphisms and approximation for Cantor systems. *Topology Appl.*, 161:178–195, 2014.
- [27] Takashi Shimomura. The construction of a completely scrambled system by graph covers. *Proc. Amer. Math. Soc.*, 144(5):2109–2120, 2016.
- [28] Takashi Shimomura. Graph covers and ergodicity for zero-dimensional systems. *Ergodic Theory Dynam. Systems*, 36(2):608–631, 2016.
- [29] Ja. G. Sinaĭ. Markov partitions and U-diffeomorphisms. *Funkcional. Anal. i Priložen*, 2(1):64–89, 1968.
- [30] Ja. G. Sinaĭ. Gibbs measures in ergodic theory. *Uspehi Mat. Nauk*, 27(4(166)):21–64, 1972.
- [31] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747–817, 1967.
- [32] Stephen Smale. Diffeomorphisms with many periodic points. In *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, pages 63–80. Princeton Univ. Press, Princeton, N.J., 1965.
- [33] P. Walters. On the pseudo-orbit tracing property and its relationship to stability. In *The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977)*, volume 668 of *Lecture Notes in Math.*, pages 231–244. Springer, Berlin, 1978.
- [34] R. F. Williams. One-dimensional non-wandering sets. *Topology*, 6:473–487, 1967.

(C. Good) UNIVERSITY OF BIRMINGHAM, SCHOOL OF MATHEMATICS, BIRMINGHAM B15 2TT, UK

E-mail address, C. Good: `c.good@bham.ac.uk`

(J. Meddaugh) UNIVERSITY OF BIRMINGHAM, SCHOOL OF MATHEMATICS, BIRMINGHAM B15 2TT, UK

E-mail address, J. Meddaugh: `j.meddaugh@bham.ac.uk`